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A class of homogeneous Einstein–Maxwell fields

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Abstract. I consider solutions of the Einstein–Maxwell field equations satisfying the conditions

$$F^{+}_{ij;k}l^k = fF^{+}_{ij}, \qquad F^{+}_{ij;k}n^k = gF^{+}_{ij}$$

where F^{+}_{ij} is the self-dual Maxwell bivector and l^i and n^i are the principal null vectors of F^{+}_{ij}. The fields are algebraically general and the most general solution is found in closed form when l^i and n^i have non-zero twist. Several twist-free solutions are also obtained. Vacuum space–times admitting a test Maxwell field are also considered. In all cases the metric admits a three-parameter group of motions acting transitively on three-dimensional orbits. Fields satisfying the dual conditions with l^i and n^i replaced by m^i and $\bar{m}^i$ are also investigated and similar results are obtained.

1. Introduction

Recently a number of authors have considered Einstein–Maxwell fields in which the principal null tetrad $(l^i, n^i, m^i, \bar{m}^i)$ of the Maxwell bivector is parallel propagated along its two real vectors l^i and n^i. The solution for the case where l^i and n^i are twist-free has been obtained by Tariq and Tupper (1974, 1975). The most general divergence-free solutions have been found by McLenaghan and Tariq (1975) and independently by Tupper (1976). The author (Barnes 1976) has considered Maxwell fields satisfying the dual conditions that the null tetrad is parallel propagated along its two complex vectors m^i and $\bar{m}^i$. The solution for the case where m^i and $\bar{m}^i$ are twist-free was found in closed form. Subsequently it has been shown by McLenaghan and Tariq (1976) and Barnes (1977) that the three solutions mentioned above are in fact the only parallel propagated solutions.

The conditions that the null tetrad is parallel propagated along l^i and n^i (or equivalently along any direction in the timelike eigenblade of F^{+}_{ij}) imply the following restrictions on the spin coefficients:

$$\kappa = \nu = \pi = \tau = 0 \tag{1.1a}$$

$$\epsilon = \gamma = 0. \tag{1.1b}$$

Here and in what follows I use the spin coefficient formalism and notation of Newman and Penrose (1962) (referred to below as NP). The analogous conditions for parallel propagation of the tetrad along directions lying in the space-like eigenblade of F^{+}_{ij} are

$$\sigma = \lambda = \rho = \mu = 0 \tag{1.2a}$$

$$\alpha = \beta = 0. \tag{1.2b}$$

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However, as McLenaghan and Tariq (1976) have shown it is convenient to employ a tetrad for which only (1.1*a*) or (1.2*a*) is valid. These two equations are invariant under the tetrad rescaling:

$$\tilde{l}^i = A l^i, \qquad \tilde{n}^i = A^{-1} n^i, \qquad \tilde{m}^i = e^{i\theta} m^i \tag{1.3}$$

whereas (1.1*b*) and (1.2*b*) are not. The existence of a rescaling for which (1.1*b*) (or (1.2*b*)) is valid leads to an integrability condition relating the Weyl tensor component ψ_2 and the electromagnetic field strength ϕ_1 namely: $\psi_2 + \phi_1\bar{\phi}_1 = 0$ (or $\psi_2 - \phi_1\bar{\phi}_1 = 0$ in the space-like case).

Whilst the conditions (1.1*a*) or (1.2*a*) seem quite natural and are satisfied for example by the Reissner–Nordström solution, the integrability conditions or equivalently (1.1*b*) and (1.2*b*) are restrictive and lead to solutions whose physical significance is unclear. Furthermore, the integrability conditions do not allow the electromagnetic field to vanish unless the Weyl tensor is type N.

In this paper the integrability conditions are discarded but (1.1*a*) (or (1.2*a*)) is retained. Fields satisfying either of these equations will be called weakly parallel propagated.

When the twists of l^i and n^i (respectively of m^i and $\bar{m}^i$) are non-zero the fields have many properties in common with parallel-propagated fields and in fact all the solutions can be obtained in closed form. The methods of this paper are also applicable to vacuum fields which admit a weakly parallel-propagated test electromagnetic field. The vacuum solutions of Kasner (1925) are shown to be of this type.

A somewhat curious feature is that the apparently restrictive condition that the twists of l^i and n^i are zero in fact leads to a wider class of solutions. This arises because the integrability conditions for the fields involve the twist $\rho - \bar{\rho}$ (or $\tau + \bar{\pi}$) in a crucial way and they are more easily satisfied when the twist vanishes. Two distinct classes of twist-free solutions are considered in detail in this paper, but it appears that a much wider class of twist-free solutions exist which are not amenable to the analysis used below.

2. The time-like case

The self-dual Maxwell bivector, which I assume is non-null, can be written in the form

$$F^+_{ij} = \phi_1(n_{[i}l_{j]} + m_{[i}\bar{m}_{j]}) \tag{2.1}$$

where ϕ_1 is the complex field strength and l^i, n^i, m^i and $\bar{m}^i$ form a principal null tetrad of the electromagnetic field and satisfy the usual orthonormality conditions. Equation (2.1) defines the tetrad only up to transformations of the form (1.3). For Einstein–Maxwell fields the only non-zero component of the Ricci tensor is $\phi_{11} = \phi_1\bar{\phi}_1$ whereas for a vacuum space–time the Ricci tensor is of course zero.

As equations (1.1*a*) and (2.1) are invariant under the tetrad rescaling (1.3) the scaling factors A and θ can be chosen to simplify the remaining spin coefficient equations. By a straightforward generalisation of an argument of McLenaghan and Tariq (1976) and Barnes (1977) it can be shown that a tetrad exists such that

$$\rho = \omega\mu, \qquad \sigma = \omega\lambda, \qquad \epsilon = \omega\gamma, \qquad \alpha = \beta = 0,$$
$$\psi_0 = \psi_4, \qquad \psi_1 = \psi_3 = 0, \tag{2.2}$$

and such that for any spin coefficient x

$$\delta x = \bar{\delta} x = (D + \omega \Delta) x = 0, \tag{2.3}$$

where $\omega = \pm 1$.

The remaining Ricci identities are

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \rho(\epsilon + \bar{\epsilon}) \tag{2.4a}$$

$$D\sigma = \rho(\sigma + \bar{\sigma}) - (3\epsilon - \bar{\epsilon})\sigma \tag{2.4b}$$

$$D\epsilon = -\epsilon(\epsilon + \bar{\epsilon}) + \tfrac{1}{2}\omega(\psi_2 + \phi_{11}) \tag{2.4c}$$

$$\omega\psi_2 = \rho(\rho - \bar{\rho}) + \sigma(\bar{\sigma} - \sigma) + 2\rho(\epsilon + \bar{\epsilon}) \tag{2.4d}$$

$$\omega\phi_{11} = \sigma\bar{\sigma} - \rho\bar{\rho} + 2\rho\bar{\epsilon} + 2\bar{\rho}\epsilon \tag{2.4e}$$

$$\psi_0 = \rho\bar{\sigma} - \bar{\rho}\sigma - 2(3\epsilon - \bar{\epsilon})\sigma. \tag{2.4f}$$

If the twist of l^i vanishes the above results are not necessarily valid and it is necessary to assume in addition that $\sigma\bar{\sigma} = \lambda\bar{\lambda}$ and $\psi_0 = \psi_4$ in order to derive them. Below we will only consider twist-free fields for which these additional assumptions are valid.

If the shear of l^i is zero the Weyl tensor is algebraically special. Equation (4.2) of NP reveals that the field is type D if $\rho \neq 0$ or type N if $\rho = 0$. All type D vacuum and Einstein–Maxwell fields have been found by Kinnersley (1969a, 1969b) and consequently only algebraically general solutions will be considered below.

If the commutator relations (4.4) of NP are applied to the coordinate functions x^i one can deduce that coordinates exist such that

$$l^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial u}, \qquad n^i \frac{\partial}{\partial x^i} = -\omega \frac{\partial}{\partial u} + Q^{-1} \frac{\partial}{\partial v} \tag{2.5}$$

where $Q = \exp(\int (\epsilon + \bar{\epsilon})\, du)$. One also obtains differential equations for the tetrad vector m^i from which one can deduce by an argument similar to that of Barnes (1977) that

$$(D + \epsilon + \bar{\epsilon})(\sigma - \bar{\sigma}) = 0 \tag{2.6a}$$

$$(D + \epsilon + \bar{\epsilon})(\rho - \bar{\rho} - 2\epsilon + 2\bar{\epsilon}) = 0. \tag{2.6b}$$

A lengthy argument then shows that (2.4) and (2.6) are consistent only in the following two cases:

Case 1 $\quad \rho - \bar{\rho} = 2(\epsilon - \bar{\epsilon}), \qquad \sigma = \bar{\sigma} \quad$ (2.7*a*)

Case 2 $\quad \rho - \bar{\rho} = \epsilon - \bar{\epsilon} = 0, \qquad \sigma + \bar{\sigma} = 0. \quad$ (2.7*b*)

From (2.3), (2.4), and (2.5) it follows that all the spin coefficients depend only on the coordinate u. In § 3 I assume that at least one of the spin coefficients is not constant. It turns out that the only solution ruled out by this assumption is the parallel-propagated divergence-free solution of McLenaghan and Tariq (1975).

3. The solutions for the time-like case

3.1. Case 1

On noting (2.7a) the integration of the differential equations for the vectors m^i and $\bar{m}^i$ is easily completed.

The results are

$$\sqrt{2}m^i\frac{\partial}{\partial x^i}=P^{-1}\frac{\partial}{\partial x}+\mathrm{i}R^{-1}\left(\frac{\partial}{\partial y}-2\omega Dx\frac{\partial}{\partial v}\right) \tag{3.1}$$

where

$$P=\exp\left(-\tfrac{1}{2}\int(\rho+\bar{\rho}+2\sigma)\,\mathrm{d}u\right)\qquad R=\exp\left(-\tfrac{1}{2}\int(\rho+\bar{\rho}-2\sigma)\,\mathrm{d}u\right)$$

and D is a constant which vanishes if the twists of l^i and n^i are zero. The metric takes the form

$$\mathrm{d}s^2=(2\omega Q\,\mathrm{d}v+2DQx\,\mathrm{d}y)\left(\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}t+Q\,\mathrm{d}v+2\omega DQ\,\mathrm{d}y\right)-P^2\,\mathrm{d}x^2-R^2\,\mathrm{d}y^2 \tag{3.2}$$

where $t=t(u)$ is a new coordinate introduced for later convenience.

This metric admits a three-parameter local isometry group generated by the Killing vectors

$$X_1=\frac{\partial}{\partial v},\qquad X_2=\frac{\partial}{\partial y},\qquad X_3=\frac{\partial}{\partial x}-2\omega Dy\frac{\partial}{\partial v}$$

which form a basis of a Lie algebra of Bianchi type 2 if $D\neq 0$ or Bianchi type 1 (i.e. Abelian) if $D=0$. The orbits are time-like if $\omega=+1$ and space-like if $\omega=-1$.

To complete the solution it remains to solve equations (2.4a, b, c) for ρ, σ and ϵ and to evaluate the expressions for the metric functions P, Q and R. This is greatly facilitated by noting the simple integral

$$2(\epsilon+\bar{\epsilon})=\rho+\bar{\rho}-2a\sigma \tag{3.3}$$

where a is a constant. A number of subclasses arise.

3.1.1. $D\neq 0$. For the generic case $D\neq 0$ the results are

$$\frac{\mathrm{d}t}{\mathrm{d}u}=[2ct+b(1+t^2)]\sigma \tag{3.4a}$$

$$\sigma=D\left(\frac{2ct+b(1+t^2)}{1+t^2}\right)^{1/2}F(t) \tag{3.4b}$$

$$\rho-\bar{\rho}=2\mathrm{i}\sigma\frac{2ct+b(1+t^2)}{1+t^2} \tag{3.4c}$$

$$\rho+\bar{\rho}=2\sigma\frac{c(1-t^2)+a(1+t^2)}{1+t^2} \tag{3.4d}$$

where b and $c \equiv (a^2+b^2-1)^{1/2}$ are constants and where $F(t)$ is given by

$$F(t)=\begin{cases} t^{a/c} & \text{when } b=0 \\ \exp\left(\dfrac{-2a}{b(1+t)}\right) & \text{when } b\neq 0 \text{ and } |a|=1 \\ \exp\left(\dfrac{2a}{\sqrt{(1-a^2)}}\tan^{-1}\dfrac{bt+c}{\sqrt{(1-a^2)}}\right) & \text{when } b\neq 0 \text{ and } |a|<1 \\ \left(\dfrac{bt+c-\sqrt{(a^2-1)}}{bt+c+\sqrt{(a^2-1)}}\right)^{a/\sqrt{(a^2-1)}} & \text{when } b\neq 0 \text{ and } |a|>1. \end{cases}$$

The metric functions are

$$Q=\left(\frac{2ct+b(1+t^2)}{1+t^2}\right), \qquad P=Q^{-1}F^{-(a+1)/2a}, \qquad R=Q^{-1}F^{-(a-1)/2a}. \tag{3.5}$$

The curvature tensor components can easily be calculated from (2.4d, e, f). For example the Ricci tensor is given by

$$\omega\phi_{11}=2b\sigma^2\frac{2ct+b(1+t^2)}{1+t^2}.$$

Consequently if $b=0$ the space-time is empty. The electromagnetic field strength is given by

$$\phi_1=\phi_0\,\mathrm{e}^{\mathrm{i}p}\frac{1-t^2-2\mathrm{i}t}{1+t^2}$$

where $\phi_0=\sqrt{\phi_{11}}$ for Einstein–Maxwell fields and $\phi_0=At^{1+a/\sqrt{(a^2-1)}}/(1+t^2)$ for the test-field case and where p and A are constants.

3.1.2. $D=0$. For the generic twist-free case ($D=0$) the results are

$$\frac{\mathrm{d}t}{\mathrm{d}u}=(t^2+1-a^2)\sigma \tag{3.6a}$$

$$\sigma=b(t^2+1-a^2)F(t) \tag{3.6b}$$

$$\rho=(t+a)\sigma \tag{3.6c}$$

where b is a constant and $F(t)$ is given by

$$F(t)=\begin{cases} \mathrm{e}^{-2a/t} & \text{for } |a|=1 \\ \exp\left[\dfrac{2a}{\sqrt{(1-a^2)}}\tan^{-1}\dfrac{t}{\sqrt{(1-a^2)}}\right] & \text{for } |a|<1 \\ \left(\dfrac{t-\sqrt{(a^2-1)}}{t+\sqrt{(a^2-1)}}\right)^{a/\sqrt{(a^2-1)}} & \text{for } |a|>1. \end{cases}$$

The metric functions in this case are

$$Q=(t^2+1-a^2)^{1/2} \qquad P=Q^{-1}F^{-(a+1)/2a} \qquad R=Q^{-1}F^{-(a-1)/2a}. \tag{3.7}$$

The Ricci tensor and electromagnetic field strength are

$$\phi_{11}=\sigma^2(t^2+1-a^2) \qquad \phi_1=\surd\phi_{11}\,\mathrm{e}^{\mathrm{i}p}$$

where p is a constant

3.1.3. The method used to derive the results for classes 1 and 2 above is not valid if ρ is a constant multiple of σ. In this case the twisting solutions are given by

$$\rho=\frac{-1+2\mathrm{i}D}{2t}, \qquad \sigma=\frac{E}{2t}, \qquad \epsilon=\frac{\mathrm{i}D}{2t}, \qquad \frac{\mathrm{d}t}{\mathrm{d}u}=1 \tag{3.8}$$

where $E\equiv\surd(4D^2+1)$. The functions appearing in the metric are given by

$$Q=1, \qquad P=t^{(E-1)/2}, \qquad R=t^{(E+1)/}2. \tag{3.9}$$

The Ricci tensor and electromagnetic field strength are given by

$$\phi_{11}=\frac{2D}{t^2}, \qquad \phi_1=\surd2D\,\mathrm{e}^{\mathrm{i}p}t^{-1+2\mathrm{i}D}.$$

3.1.4. The twist-free case is given by

$$\rho=\frac{-2C^2}{(3C^2+1)t}, \qquad \sigma=\frac{-2C}{(3C^2+1)t}, \qquad 2\epsilon=\frac{1-C^2}{(3C^2+1)t}, \qquad \frac{\mathrm{d}t}{\mathrm{d}u}=1 \tag{3.10}$$

$$\ln P=\frac{2(C^2+C)}{3C^2+1}\ln t, \qquad \ln Q=\frac{C^2-1}{3C^2+1}\ln t, \qquad \ln R=\frac{2(C^2-C)}{3C^2+1}\ln t, \tag{3.11}$$

where C is a constant. The Ricci tensor vanishes for this metric which is a vacuum Kasner metric.

3.2. *Case 2.*

In this case equation (2.7b) is valid. The tetrad vectors l^i and n^i are given by (2.5). The complex tetrad vector m^i and metric are given by

$$\surd2m^i\frac{\partial}{\partial x^i}=P^{-1}\left(\mathrm{e}^Z\frac{\partial}{\partial x}+\mathrm{i}\,\mathrm{e}^{-Z}\frac{\partial}{\partial y}\right) \tag{3.12a}$$

$$\mathrm{d}s^2=2Q\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}t\,\mathrm{d}v+2\omega Q^2\,\mathrm{d}v^2-P^2(\mathrm{e}^{-2Z}\,\mathrm{d}x^2+\mathrm{e}^{2Z}\,\mathrm{d}y^2) \tag{3.12b}$$

where $P=\exp(-\int\rho\,\mathrm{d}u)$, $Z=2\Sigma_0v-\mathrm{i}\int\sigma\,\mathrm{d}u$ and Σ_0 is a constant. This metric like (3.2) admits a three-parameter isometry group with three-dimensional orbits generated by the Killing vectors

$$X_1=\frac{\partial}{\partial u}, \qquad X_2=\frac{\partial}{\partial y}, \qquad X_3=\frac{\partial}{\partial v}-2\Sigma_0\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)$$

which form a basis of a Lie algebra of Bianchi type 6. The orbits are time-like if $\omega=+1$ and space-like if $\omega=-1$.

I have been unable to integrate (2.4) when $\phi_{11} \neq 0$. However, the general vacuum solution is

$$\frac{dt}{du} = -\Sigma_0 t \exp(\tfrac{1}{2}t^{-1})[t^3(t-4)^5]^{1/8} \tag{3.13a}$$

$$\epsilon = \Sigma_0 \exp(\tfrac{1}{2}t^{-1})[t^5(t-4)^3]^{-1/8} \tag{3.13b}$$

$$\rho = t\epsilon \tag{3.13c}$$

$$\sigma^2 = t(t-4)\epsilon^2 \tag{3.13d}$$

$$Q = \exp(-\tfrac{1}{2}t^{-1})[(t-4)/t]^{-1/8} \tag{3.14a}$$

$$P = [(t-4)/t]^{1/4} \tag{3.14b}$$

$$Z = 2\Sigma_0 v - \tfrac{1}{2}[(t-4)/t]^{1/2} \tag{3.14c}$$

4. The space-like case

In this section I consider fields which satisfy (1.2a) or equivalently

$$F^{+}_{ij;k} m^k = f F^{+}_{ij}, \qquad F^{+}_{ij;k} \bar{m}^k = g F^{+}_{ij}.$$

Most of the results may be obtained from those in the time-like case by formally replacing l^i by m^i and n^i by $-\bar{m}^i$ and vice versa as in Barnes (1976). Corresponding to (2.2) and (2.3) we have

$$\tau = \pi, \qquad \alpha = \beta, \qquad \kappa = \omega\nu, \qquad \psi_0 = \omega\psi_4, \qquad \epsilon = \gamma = 0 \tag{4.1}$$

where $\omega = \pm 1$ and for any spin coefficient x

$$Dx = \Delta x = (\delta + \bar{\delta})x = 0. \tag{4.2}$$

As in time-like case difficulties arise in deriving (4.1) and (4.2) when the twist of m^i is zero ($\tau + \bar{\pi} = 0$). However, we will only consider those twist-free solutions for which (4.1) and (4.2) are valid. If $\kappa = \tau = 0$ the metric is Petrov type N. However, if $\kappa = 0$ but $\tau \neq 0$ we cannot conclude that the field is type D in the time-like case. It seems probable that type 2 fields exist in the space-like case. The above remark shows that some care must be exercised when dualising proofs. As in the time-like case I now restrict attention to algebraically general fields.

Coordinates exist such that

$$\sqrt{2} m^i \frac{\partial}{\partial x^i} = P^{-1}\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y} \tag{4.3}$$

where $P = \exp(\mathrm{i}\sqrt{2}\int(\alpha - \bar{\alpha})\,dy)$ and all spin coefficients are functions of the coordinate y only. Just as in the time-like case two classes of solution arise:

Case 1 $\qquad \kappa + \bar{\kappa} = 0, \qquad \tau + \bar{\tau} = 2(\alpha + \bar{\alpha}) \qquad$ (4.4a)

Case 2 $\qquad \kappa - \bar{\kappa} = 0, \qquad \tau + \bar{\tau} = \alpha + \bar{\alpha} = 0. \qquad$ (4.4b)

4.1. *Case 1*

The real null vectors l^i and n^i are given by

$$l^i \frac{\partial}{\partial x^i} = e^A \left[F_\omega(B) \frac{\partial}{\partial u} + G_\omega(B) \left(\frac{\partial}{\partial v} + 2\sqrt{2} Du \frac{\partial}{\partial x} \right) \right] \tag{4.5a}$$

$$n^i \frac{\partial}{\partial x^i} = e^A \left[\omega G_\omega(B) \frac{\partial}{\partial u} + F_\omega(B) \left(\frac{\partial}{\partial v} + 2\sqrt{2} Du \frac{\partial}{\partial x} \right) \right] \tag{4.5b}$$

where

$$A = \frac{1}{2i} \int (\tau - \bar{\tau}) \sqrt{2} dy, \qquad B = i \int \kappa \sqrt{2}\, dy$$

and D is a constant which vanishes if m^i is twist-free (i.e. $\tau + \bar{\tau} = 0$). For $\omega = +1, F_\omega(B) = \cosh B$ and $G_\omega(B) = \sinh B$ whereas for $\omega = -1, F_\omega(B) = \cos B$ and $G_\omega(B) = \sin B$. The metric is

$$ds^2 = -e^{-2A} G_\omega(2B)(du^2 + \omega\, dv^2) + 2\, e^{-2A} F_\omega(2B)\, du\, dv - P^2(dx - 2\sqrt{2} Du\, dv)^2 - \left(\frac{dy}{dz} \right)^2 dz^2 \tag{4.6}$$

where $z = z(y)$ is a new coordinate introduced for later convenience. This metric too admits a three-parameter isometry group generated by the Killing vectors

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial v}, \qquad X_3 = \frac{\partial}{\partial u} + 2\sqrt{2} Dv \frac{\partial}{\partial x}. \tag{4.7}$$

The Lie algebra is Bianchi type 2 if $D \neq 0$ and type 1 if $D = 0$. The orbits are time-like hypersurfaces.

Corresponding to (3.3) the Ricci identities admit the simple integral:

$$2(\alpha - \bar{\alpha}) = \tau - \bar{\tau} - 2a\kappa.$$

A number of subclasses of solution arise as in the time-like case 1 solutions.

4.1.1.

$$\frac{1}{\sqrt{2}} \frac{dz}{dy} = -i\kappa[2cz + b(1+z^2)] \tag{4.8a}$$

$$\kappa = iD \left(\frac{2cz + b(1+z^2)}{1+z^2} \right)^{1/2} F(z) \tag{4.8b}$$

$$\frac{1}{2}(\tau - \bar{\tau}) = \frac{[c(1-z^2) + a(1+z^2)]\kappa}{1+z^2} \tag{4.8c}$$

$$\frac{1}{2}(\tau + \bar{\tau}) = \frac{-[2cz + b(1+z^2)]i\kappa}{1+z^2} \tag{4.8d}$$

where

$$F(z) = \begin{cases} z^{a/\sqrt{(a^2-\omega)}} & \text{for } b = 0 \\ \left(\dfrac{bz + c - \sqrt{(a^2-\omega)}}{bz + c + \sqrt{(a^2-\omega)}} \right)^{a/\sqrt{(a^2-\omega)}} & \text{for } b \neq 0 \end{cases}$$

where b and $c \equiv \sqrt{(a^2 + b^2 - \omega)}$ are real constants.

The Ricci tensor is given by

$$\phi_{11} = 2b\kappa^2\left(\frac{2cz + b(1+z^2)}{1+z^2}\right)$$

and the space-time is empty if $b = 0$. The electromagnetic field strength is given by

$$\phi_1 = \sqrt{\phi_{11}}\, e^{ip}\, \frac{1 - z^2 - 2iz}{1+z^2}$$

for the Einstein–Maxwell case and by

$$\phi_1 = \phi_0\, e^{ip} z^{1+a/\sqrt{(a^2-\omega)}}\, \frac{1 - z^2 - 2iz}{(1+z^2)^2}$$

for the test-field case. The functions P, A and B which appear in the metric are given by

$$P = \left(\frac{2cz + b(1+z^2)}{1+z^2}\right)^{-1/2} \qquad A = \frac{1}{2}\ln F - \ln P, \qquad B = \frac{1}{2a}\ln F. \tag{4.9}$$

4.1.2. The generic twist-free solutions are given by

$$\frac{1}{\sqrt{2}}\frac{dz}{dy} = -i\kappa(z^2 + \omega - a^2) \tag{4.10a}$$

$$\kappa = ib(z^2 + \omega - a^2)^{1/2} F(z) \tag{4.10b}$$

$$\tfrac{1}{2}(\tau - \bar{\tau}) = (z + a)\kappa \tag{4.10c}$$

where a and b are constants and $F(z) = \{[z - \sqrt{(a^2 - \omega)}]/[z + \sqrt{(a^2-\omega)}]\}^{a/\sqrt{(a^2-\omega)}}$. The metric functions in this case are

$$P = (z^2 + \omega - a^2)^{-1/2}, \qquad A = \frac{1}{2}\ln F(z) - \ln P, \qquad B = \frac{1}{2a}\ln F(z) \tag{4.11}$$

The Ricci tensor and electromagnetic field strength are

$$\phi_{11} = \kappa^2(z^2 + \omega - a^2) \qquad \phi_1 = \sqrt{\phi_{11}}\, e^{ip}.$$

There are no twisting solutions for which τ is a constant multiple of κ, but a twist-free analogue of the solution given in (3.10,11) does exist.

4.1.3. This field like its timelike analogue is of Kasner vacuum type and the metric is diagonalisable if $\omega = +1$. The spin coefficients and metric functions are given by

$$\kappa = \frac{-2iC}{(3C^2+\omega)\sqrt{2}z}, \qquad \tau = \frac{-2iC}{(3C^2+\omega)\sqrt{2}z},$$

$$2\alpha = \frac{-i(C^2 - \omega)}{(3C^2+\omega)\sqrt{2}z}, \qquad \frac{dz}{dy} = 1 \tag{4.12}$$

$$\ln P = \frac{C^2-\omega}{3C^2+\omega}\ln z, \qquad A = \frac{-2C^2}{3C^2+\omega}\ln z, \qquad B = \frac{-2C}{3C^2+\omega}\ln z. \tag{4.13}$$

4.1.4. A twist-free vacuum field of Kasner type also exists in which all the spin coefficients are constants:

$$\kappa = \sqrt{(\tfrac{3}{2})}\mathrm{i}C \qquad \tau = \mathrm{i}C/\sqrt{2} \qquad \alpha = \mathrm{i}C/\sqrt{2}, \qquad \mathrm{d}z/\mathrm{d}y = 1 \tag{4.14}$$

$$P = \mathrm{e}^{2Cz}, \qquad A = Cz, \qquad B = \sqrt{3}Cz \tag{4.15}$$

where C is a real constant and in (4.5) and (4.6) $\omega = -1$. The metric admits a four-parameter transitive group of motions generated by the Killing vectors X_1, X_2 and X_3 of (4.7) together with

$$X_4 = \frac{\partial}{\partial z} - 2Cx\frac{\partial}{\partial x} - C(u - \sqrt{3}v)\frac{\partial}{\partial u} + C(v + \sqrt{3}u)\frac{\partial}{\partial v}.$$

The complete Lie algebra is type 4_4 in the classification of four-dimensional algebras given in Petrov (1969). It is the only solution which arises from the assumption that all the spin coefficients are constants. It is perhaps worth noting that this metric is not dual to the parallel-propagated field of McLenaghan and Tariq (1975) which is the only field in the time-like case which has all its spin coefficients constant.

4.2. Case 2

For this class of solutions (4.4*b*) is valid. The complex tetrad vector m^i is still given by equation (4.3). The real null vectors and the metric take the following form:

$$l^i\frac{\partial}{\partial x^i} = A^{-1}\left(F_{-\omega}(\sqrt{2}K_0x)\frac{\partial}{\partial u} + G_{-\omega}(\sqrt{2}K_0x)\frac{\partial}{\partial v}\right) \tag{4.16a}$$

$$n^i\frac{\partial}{\partial x^i} = A^{-1}\left(-\omega G_{-\omega}(\sqrt{2}K_0x)\frac{\partial}{\partial u} + F_{-\omega}(\sqrt{2}K_0x)\frac{\partial}{\partial v}\right) \tag{4.16b}$$

$$\begin{aligned} \mathrm{d}s^2 = {} & -A^2G_{-\omega}(2\sqrt{2}K_0x)(\mathrm{d}u^2 - \omega\,\mathrm{d}v^2) \\ & + 2A^2F_{-\omega}(2\sqrt{2}K_0x)\,\mathrm{d}u\,\mathrm{d}v - P^2\,\mathrm{d}x^2 - \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2\mathrm{d}z^2 \end{aligned} \tag{4.17}$$

where

$$A = \exp\left(\mathrm{i}\int \tau\sqrt{2}\,\mathrm{d}y\right), \qquad P = \exp\left(-2\mathrm{i}\int \alpha\sqrt{2}\,\mathrm{d}y\right), \qquad \kappa = K_0/P$$

and where F and G are defined as in equation (4.5).

This metric admits a three-parameter isometry group generated by the Killing vectors

$$X_1 = \frac{\partial}{\partial u}, \qquad X_2 = \frac{\partial}{\partial v}, \qquad X_3 = \frac{\partial}{\partial x} + K_0\left(u\frac{\partial}{\partial v} - \omega v\frac{\partial}{\partial u}\right).$$

The Lie algebra is Bianchi type 6 if $\omega = -1$ or Bianchi type 7 if $\omega = +1$. The orbits are three-dimensional time-like hypersurfaces.

I have again been unable to integrate the Ricci identities for α, κ and τ except for the vacuum case. The general vacuum solution is given by

$$\tau = \alpha z, \qquad \kappa = K_0\exp(\tfrac{1}{2}z^{-1})\left(\frac{\omega(z-4)}{z}\right)^{1/8} \tag{4.18a}$$

$$\alpha = \mathrm{i}K_0 \exp(\tfrac{1}{2}z^{-1})[\omega z^5(z-4)^3]^{-1/8}, \qquad \frac{1}{\sqrt{2}}\frac{\mathrm{d}z}{\mathrm{d}y} = -\omega K_0 z \exp(\tfrac{1}{2}z^{-1})[\omega z^3(z-4)^5]^{1/8} \tag{4.18b}$$

$$A = \left(\frac{\omega(z-4)}{z}\right)^{1/4}, \qquad P = \exp\left(\frac{1}{2}z^{-1}\right)\left(\frac{\omega(z-4)}{z}\right)^{1/8}. \tag{4.19}$$

5. Summary and conclusions

Weakly parallel-propagated fields have been considered and the general solutions with non-zero twist have been found explicitly for both the time-like and space-like cases. In addition two distinct classes of twist-free solution have been obtained. In all cases the metric admits a three-parameter group of motions with three-dimensional orbits. The solutions are considerably more general than the parallel-propagated solutions already known. The solutions with non-zero twist involve three arbitrary constants (four if one includes the complexion p of the electromagnetic field) compared with one arbitrary constant in the solution of McLenaghan and Tariq (1975). The twist-free solutions involve two arbitrary constants compared with none in the twist-free parallel propagated solutions of Tariq and Tupper (1975) and Barnes (1976).

From a physical viewpoint the most important solutions are those in the time-like case with $\omega = -1$. The orbits of the isometry group are space-like in this case and the solutions represent homogeneous vacuum or electrovac cosmological models. The metric (3.2) can be written in a more familiar form without cross-terms involving $\mathrm{d}u$ by means of the coordinate transformation $\mathrm{d}z = \mathrm{d}v + \frac{1}{2}Q^{-1}\,\mathrm{d}u$. The coordinate u is (apart from a constant factor) a proper-time coordinate whose level surfaces are the surfaces of homogeneity. However, it seems more convenient to retain the coordinate t introduced in (3.2) as the functions appearing in the metric can be written down explicitly whereas they can usually only be written as implicit functions of u. The twisting solutions are Bianchi type 2 and the twist-free solutions are Bianchi type 1 or 6.

The analysis used in this paper can be extended to include a perfect fluid flow which is aligned with the electromagnetic field. A further paper is in preparation in which the perfect fluid fields are presented. The dynamics of the solutions will also be investigated.

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